Hofstadter Problem on the Honeycomb and Triangular Lattices: Bethe Ansatz Solution

M. Kohmoto [a] [*] and A. Sedrakyan[a,b] [†]

[a] *Institute of Solid State Physics, University of Tokyo,*
Kashiwanoha 5-1-5, Kashiwa-shi, Chiba, 277-8581, Japan
[b] *Yerevan Physics Institute,Br.Alikhanian str.2, Yerevan 36, Armenia*
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We consider Bloch electrons on the honeycomb lattice under a uniform magnetic field with $2\pi p/q$ flux per cell. It is shown that the problem factorizes to two triangular lattices. Treating magnetic translations as Heisenberg-Weyl group and by the use of its irreducible representation on the space of theta functions, we find a nested set of Bethe equations, which determine the eigenstates and energy spectrum. The Bethe equations have simple form which allows to consider them further in the limit $p, q \to \infty$ by the technique of Thermodynamic Bethe Ansatz and analyze Hofstadter problem for the irrational flux.

I. INTRODUCTION

The quantum Hall effect (QHE) on graphite thin films has been observed recently[1,2] . It has the honeycomb lattice with one π electron per lattice site. The only Fermi levels ($E_F = 0$) are two points of the Brillouin zone where the conduction band and the valence bands touch forming cones. Low energy excitations have a linear spectrum as for the massless relativistic Dirac particles in 2+1 dimensional space-time.[3,4,5]. In[6,7,8] the Hall effect on graphene on the basis of Dirac electrons was investigated.

The problem of electrons in two dimensional periodic potential in a magnetic field has been attracted a lot of attention due to the unexpected nature of the commensurability and frustrations.[9,10,11,12,13,14,15,16,17,18,19,21,22]. It has let the existence of Cantor set energy spectra, and the wavefunctions strongly depend on wether the flux is commensurate or incommensurate with the lattice. It was observed[12] that when the magnetic flux corresponding to unit cell of the periodic structure of the system is incommensurate with the potential, the one particle spectrum exhibits multifractal behavior like the Cantor set (see also[16]). Then, after discovering the topological character of the conductance in the model by linking it with the Chern class in the linear bundle of the Bloch wave function[13,14], its possible connection with quantum Hall effect become evident.

Usually a solid with well localized atomic orbits is modelled by the lattice and the investigations were carried out by the use of a tight-binding Hamiltonian.. The presence of zero modes was pointed out in[15] and the model with nearest-neighbor and next-to-nearest-neighbor hopping was studied in[17]. Wiegmann and Zabrodin[19] have observed the presence of a hidden quantum group in the Hofstadter problem on the regular lattice and initiated studies[20,21,22] of the integrable structure in it. The approach allowed to express the spectrum and the Bloch wave function at the mid-band points as solutions of the Bethe equations typical for completely integrable quantum systems. In the article[20] the problem have been analyzed by the use of cyclic representations of quantum groups at any momenta, but the explicit form of the Bethe equations was not found(besides the midband point).

In[21] authors have used the fact that magnetic translations in Hofstadter problem are forming Heisenberg-Weyl group and analyzed its irreducible representations on the basis of theta functions[23]. This approach allowed us to find explicit form of the Bethe equations for the spectrum and the wave-functions not only at the midband point, but for any momenta.

Motivated by the discovered unconventional Hall effect in graphene we study in this article the problem of Bloch electrons in a uniform magnetic field with flux $2\pi p/q$ per cell on the honeycomb lattice. We show that the problem possesses a chiral factorization and it reduces to two (R and L) triangular sublattices. Following the line of[21], we look for the solution of the Schrödinger equation for energy and eigenfunctions in the space of irreducible representation of the Heisenberg-Weyl group, namely, in the space of theta functions with characteristics. By the use of nesting procedure, similar to the one appeared in Algebraic Bethe Ansatz for some of integrable models (see for example[24,25,26]), we have transformed Schrödinger equation into the simple set of nested Bethe equations for the spectral parameters. We have found a spectrum of the model and eigenstates as a function of the solution of the nested Bethe Anzats equations, which depend on momenta $\vec{k}$.

The form of the nested Bethe equations allow to investigate them in thermodynamic limit $q \to \infty$ by the technique developed by Takahashi[27] and Gaudin[28] for integrable models and called Thermodynamic Bethe Ansatz. This limit is important for the most interesting irrational flux case of the Hofstadter problem since q gives the number of bands and we face a Cantor like behavior of the spectrum. Irrational number can be approached by the rationales p/q with $p, q \to \infty$.

II.   ELECTRONS ON THE 2D HONEYCOMB LATTICE IN A UNIFORM MAGNETIC FILED

Let us start with formulation of Hofstadter problem on the honeycomb lattice $\mathcal{L}$ (see FIG.1). The honeycomb lattice possesses chiral factorization, namely one can consider $\mathcal{L} = \mathcal{L}^L + \mathcal{L}^R$ as a sum of two triangular lattices $\mathcal{L}^L$ and $\mathcal{L}^R$, each of which is isomorphic to the dual of the honeycomb lattice. Sites of the lattices $\mathcal{L}^L$ and $\mathcal{L}^R$ on the Fig.1 are marked with black and yellow dots respectively. The links of the honeycomb lattice are connecting the sites on $\mathcal{L}^R$ with the three nearest neighbor sites on $\mathcal{L}^R$.
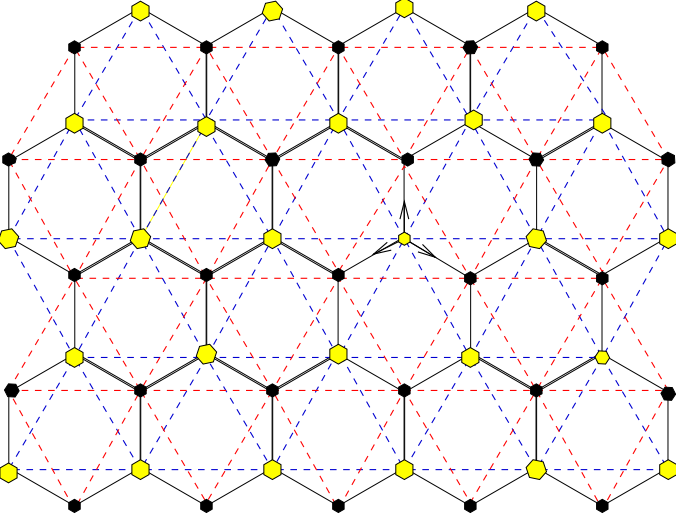

FIG. 1: The representation of the honeycomb lattice as a joint of triangular $\mathcal{L}^L$ (red) and $\mathcal{L}^R$ (blue) lattices.

The Hamiltonian is given by

$$
H = \sum_{\vec{n}\in\mathcal{L}^R,\sigma=1,2,3} \left(t_{\vec{n},\vec{n}+\vec{e}_\sigma} e^{iA_{\vec{n},\vec{n}+\vec{e}_\sigma}} c_{\vec{n}}^+ d_{\vec{n}+\vec{e}_\sigma} \right.
$$
$$
\left. + \ t_{\vec{n}+\vec{e}_\sigma,\vec{n}} e^{iA_{\vec{n}+\vec{e}_\sigma,\vec{n}}} d_{\vec{n}+\vec{e}_\sigma}^+ c_{\vec{n}} \right), \tag{1}
$$

where $c_{\vec{n}}, c_{\vec{n}}^+$ and $d_{\vec{n}+\vec{e}_\sigma}, d_{\vec{n}+\vec{e}_\sigma}^+$ are the annihilation and creation operators of electrons at the sites $\vec{n}$ and $\vec{n} + \vec{e}_\sigma$ of the triangular lattices $\mathcal{L}^R$ and $\mathcal{L}^L$ respectively. $A_{\vec{n},\vec{n}+\vec{e}_\sigma} = -A_{\vec{n}+\vec{e}_\sigma,\vec{n}}$ is the vector potential of a magnetic field with the strength perpendicular to the plane of the lattice and $t_{\vec{n},\vec{m}}$ are hoping amplitudes between the nearest neighbors. We choose them as t_1, t_2 and t_3 according to directions of the vectors $\vec{e}_\sigma$ (see Fig.2). For the homogeneous model $t_1 = t_2 = t_3$.

In the following we consider the diagonalization problem of (1) in one-particle sector. In this sector the action of the translation operator by vectors $\pm\vec{e}_\sigma$ can be written as

$$
S_{\vec{e}_\sigma} = \sum_{\vec{n}\in\mathcal{L}^R} |\vec{n}\rangle\langle\vec{n}+\vec{e}_\sigma|, \quad S_{-\vec{e}_\sigma} = \sum_{\vec{n}\in\mathcal{L}^R} |\vec{n}+\vec{e}_\sigma\rangle\langle\vec{n}|, \tag{2}
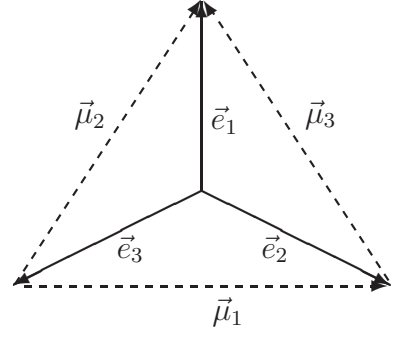$$

FIG. 2: Hopping vectors on the honeycomb ($\vec{e}_\sigma$) and triangular ($\vec{\mu}_\sigma$) lattices

where the standard bra- and ket- vectors are used: $|\vec{n}\rangle = c_{\vec{n}}^+ |vac\rangle$, $\langle\vec{n}| = \langle vac|c_{\vec{n}}$ and $|\vec{n}+\vec{e}_\sigma\rangle = d_{\vec{n}+\vec{e}_\sigma}^+ |vac\rangle$, $\langle\vec{n}+\vec{e}_\sigma| = \langle vac|d_{\vec{n}+\vec{e}_\sigma}$.

We consider the case when the magnetic flux per hexagon cell,

$$
\exp(i\Phi) = \prod_{hexagon} \exp(iA_{\vec{n},\vec{m}}), \tag{3}
$$

is rational: $\Phi = 2\pi\frac{p}{q}$, where p and q are mutually prime integers. The product in (3) is performed on anticlockwise direction.

In terms of bra- and ket- vectors the Hamiltonian (1) can be written as

$$
H = \sum_{\substack{\vec{n}\in\mathcal{L}^R,\\\sigma=1,2,3}} \left(t_{\vec{n},\vec{n}+\vec{e}_\sigma} e^{iA_{\vec{n},\vec{n}+\vec{e}_\sigma}} |\vec{n}\rangle\langle\vec{n}+\vec{e}_\sigma| \right.
$$
$$
\left. + \ t_{\vec{n}+\vec{e}_\sigma,\vec{n}} e^{iA_{\vec{n}+\vec{e}_\sigma,\vec{n}}} |\vec{n}+\vec{e}_\sigma\rangle\langle\vec{n}| \right) + h.c. \tag{4}
$$

Let us now consider the square of the Hamiltonian (4). It is straightforward to see from the expression (4) that H^2 contain only hopping terms within triangular lattices $\mathcal{L}^R$ and $\mathcal{L}^L$, namely

$$
H^2 = \mathcal{H}_R + \mathcal{H}_L, \tag{5}
$$

where

$$
\mathcal{H}_{R(L)} = \sum_{\substack{\vec{n}\in\mathcal{L}^R(\mathcal{L}^L),\\\sigma=1,2,3}} \left(\tilde{t}_{\vec{n},\vec{n}+\vec{\mu}_\sigma}^{R(L)} e^{iA_{\vec{n},\vec{n}+\vec{\mu}_\sigma}} |\vec{n}\rangle\langle\vec{n}+\vec{\mu}_\sigma| + h.c. \right)
$$
$$
+ \sum_{\vec{n}\in\mathcal{L}^R(\mathcal{L}^L)} \tilde{t}_{\vec{n},\vec{n}} |\vec{n}\rangle\langle\vec{n}|, \tag{6}
$$

with (see FIG. 2)

$$
\vec{\mu}_1 = \vec{e}_2 - \vec{e}_3, \qquad \vec{\mu}_2 = \vec{e}_1 - \vec{e}_3,
$$
$$
\vec{\mu}_3 = \vec{e}_1 - \vec{e}_2, \tag{7}
$$

and

$$
\tilde{t}_{\vec{n},\vec{n}\pm\vec{\mu}_1}^R = e^{\mp i\frac{\Phi}{6}} t_2 t_3, \qquad \tilde{t}_{\vec{n},\vec{n}\pm\vec{\mu}_2}^R = e^{\pm i\frac{\Phi}{6}} t_1 t_3,
$$
$$
\tilde{t}_{\vec{n},\vec{n}\pm\vec{\mu}_3}^R = e^{\mp i\frac{\Phi}{6}} t_1 t_2, \qquad \tilde{t}_{\vec{n},\vec{n}} = t_1^2 + t_2^2 + t_3^2
$$
$$
\tilde{t}^L = \left(\tilde{t}^R\right)^* \tag{8}
$$

This decomposition reveal the hidden chiral structure of the Hofstadter problem on the honeycomb lattice reducing it to the problem on the triangular lattice. Therefore, in the remaining part of this article, we will consider only the triangular lattice.

The problem of Bloch electrons on the triangular lattice and in uniform magnetic filed was considered recently in[29].

III. ELECTRONS ON THE TRIANGULAR LATTICE AND IN UNIFORM MAGNETIC FIELD

Denote lattice sites by vectors $\vec{n} = n_1\vec{\mu}_1 + n_2\vec{\mu}_2$. Taking different gauges one can obtain various equivalent forms of (5). In the following we will use the Landau gauge, which is

$$A_{\pm\vec{\mu}_1}(\vec{n}) = A_{\vec{n},\vec{n}\pm\vec{\mu}_1} = 0,$$
$$A_{\pm\vec{\mu}_2}(\vec{n}) = A_{\vec{n},\vec{n}\pm\vec{\mu}_2} = \pm\Phi n_1. \quad (9)$$

Then the Hamiltonian (5) is invariant under translations $S^q_{\pm\vec{\mu}_1}$, $S_{\pm\vec{\mu}_2}$ and $S^q_{\pm\vec{\mu}_3}$:

$$[\mathcal{H}, S^q_{\pm\vec{\mu}_1}] = [\mathcal{H}, S_{\pm\vec{\mu}_2}] = [\mathcal{H}, S^q_{\pm\vec{\mu}_3}] = 0.$$

So, the problem of diagonalization of $\mathcal{H}$ reduced to its diagonalization on each eigenspace of $S^q_{\pm\vec{\mu}_1}$, $S_{\pm\vec{\mu}_2}$ and $S^q_{\pm\vec{\mu}_3}$. The latter is a q-dimensional space $\Psi(\vec{k})$, spanned by Bloch wave functions

$$\psi_n(\vec{k}) = \sum_{\substack{\vec{n}_1 = n\,\mathrm{mod}\,q \\ \vec{n}_2}} e^{-i\vec{k}\vec{n}}|\vec{n}\rangle,$$

which satisfy

$$\psi_n(\vec{k}) = \psi_{n+q}(\vec{k})$$
$$S_{\pm\vec{\mu}_1}\psi_n(\vec{k}) = e^{\mp ik_1}\psi_{n\mp1}(\vec{k}) \quad (10)$$
$$S_{\pm\vec{\mu}_2}\psi_n(\vec{k}) = e^{\mp ik_2}\psi_n(\vec{k})$$
$$S_{\pm\vec{\mu}_3}\psi_n(\vec{k}) = e^{\mp i(k_2-k_1)}\psi_{n\pm1}(\vec{k}).$$

Let us define also the generators of magnetic translations by

$$T_{\pm\vec{\mu}_\sigma} = \sum_{\vec{n}} e^{iA_{\vec{n},\vec{n}\pm\vec{\mu}_\sigma}}|\vec{n}\rangle\langle\vec{n}\pm\vec{\mu}_\sigma|, \quad \sigma = 1,2,3. \quad (11)$$

It is easy to find out, that in Landau gauge (9)

$$T_{\pm\vec{\mu}_3} = e^{i\Phi/2}T_{\mp\vec{\mu}_1}T_{\pm\vec{\mu}_2}, \quad (12)$$

since the area of the triangles in honeycomb lattice (Fig.1) is the half of the area of the hexagon.

The generators $T_{\pm\vec{\mu}_1}$ and $T_{\pm\vec{\mu}_2}$ satisfy the following commutation relations

$$T_{\pm\vec{\mu}_1}T_{\pm\vec{\mu}_2} = \bar{q}^2 T_{\pm\vec{\mu}_2}T_{\pm\vec{\mu}_1}, \quad (13)$$

and form Heisenberg-Weyl group. Here we used the notation

$$\bar{q} = \exp\left(i\frac{\Phi}{2}\right) = \exp\left(\pi i\frac{p}{q}\right).$$

Note, that is any gauge $S^q_{\vec{\mu}_1,\vec{\mu}_2} = T^q_{\vec{\mu}_1,\vec{\mu}_2}$ (in the Landau gauge moreover $S_{\vec{\mu}_1} = T_{\vec{\mu}_1}$).

The action of magnetic translations $T_{\pm\vec{\mu}_1}$, $T_{\pm\vec{\mu}_2}$ on Bloch functions $\psi_n(\vec{k})$ has the following form:

$$T_{\pm\vec{\mu}_1}\psi_n(\vec{k}) = e^{\mp ik_1}\psi_{n\mp1}(\vec{k})$$
$$T_{\pm\vec{\mu}_2}\psi_n(\vec{k}) = e^{\mp ik_2 \pm in\Phi}\psi_n(\vec{k}) \quad (14)$$
$$T_{\pm\vec{\mu}_3}\psi_n(\vec{k}) = e^{\mp i(k_2-k_1)\pm i(n\pm1/2)\Phi}\psi_{n\pm1}(\vec{k}).$$

In case of translational invariant distribution of hopping parameters $\tilde{t}_{\pm\sigma} = \tilde{t}_{\vec{n},\vec{n}\pm\vec{\mu}_\sigma}$, $\sigma = 1,2,3$, and $\tilde{t} = \tilde{t}_{\vec{n},\vec{n}}$, the Hamiltonian (1) can be written in terms of the generators of magnetic translations by

$$\mathcal{H} = \sum_{\sigma=1,2,3}(\tilde{t}_\sigma T_{\vec{\mu}_\sigma} + \tilde{t}_{-\sigma}T_{-\vec{\mu}_\sigma}) + \sum_{\vec{n}}\tilde{t}|\vec{n}\rangle\langle\vec{n}|. \quad (15)$$

Note that on $\Psi(\vec{k})$ the q-th power of magnetic translations are scalars:

$$T^q_{\pm\vec{\mu}_1} = e^{\mp iqk_1}\cdot\mathrm{id}, \qquad T^q_{\pm\vec{\mu}_2} = e^{\mp iqk_2}\cdot\mathrm{id},$$
$$T^q_{\pm\vec{\mu}_3} = e^{\mp iq(k_1-k_2)}\cdot\mathrm{id} \quad (16)$$

In order to solve the Schrödinger of equation $\mathcal{H}\psi = \mathcal{E}\psi$ analiticaly we will follow the technique developed in[21] for the Hofstadter problem on the regular lattice.

IV. HEISENBERG-WEYL GROUP. REDUCTION OF THE PROBLEM TO THE BETHE EQUATIONS

Let us recall the representation of Heisenberg-Weyl group (13) on the space of complex functions[23]. It can be constructed in the following way. Define actions $\mathcal{S}_b$ and $\mathcal{T}_a$, $a,b \in Z$ on the space of analytic functions on the complex plain as

$$\mathcal{S}_b f(z) = f(z+b), \quad (17)$$
$$\mathcal{T}_a(\tau)f(z) = \exp(\pi ia^2\tau + 2\pi iaz)f(z+a\tau),$$

for some $\tau \in C$. Then

$$\mathcal{S}_a\mathcal{S}_b = \mathcal{S}_{a+b}, \mathcal{T}_a(\tau)\mathcal{T}_b(\tau) = \mathcal{T}_{a+b}(\tau)$$
$$\mathcal{S}_a\mathcal{T}_b(\tau) = \exp(2\pi iab)\mathcal{T}_b(\tau)\mathcal{S}_a. \quad (18)$$

Consider the space of theta functions with characteristics $\Theta^{(q)}(z,\tau)$ which are invariant with respect to the subgroup generated by $\mathcal{S}_{\pm q}$ and $\mathcal{T}_{\pm1}(\tau)$. They form a q-dimensional space $\mathrm{Th}_q(\tau)$ and have in the fundamental domain with vertices $(0,\tau,q,q+\tau)$ precisely q zeroes.

The space $\text{Th}_q(\tau)$ forms an irreducible representation of Heisenberg-Weyl group (13) generated by

$$T_{\pm\vec{\mu}_1} = \bar{\alpha}_\pm \mathcal{T}_{\mp\frac{1}{q}}(\tau), \quad T_{\pm\vec{\mu}_2} = \bar{\beta}_\pm \mathcal{S}_{\pm p} = \bar{\beta}_\pm \mathcal{S}^p_{\pm 1},$$

$$T_{\pm\vec{\mu}_3} = \bar{\alpha}_\mp \bar{\beta}_\pm e^{i\Phi/2} \mathcal{T}_{\pm\frac{1}{q}}(\tau) \mathcal{S}_{\pm p}. \quad (19)$$

One can choose the basis $\Theta_{a,0}(z,\tau)$ of Th_q as follows:

$$\Theta_{a,0}(z,\tau) = \left[\mathcal{T}_a(\tau)\Theta \right](z,\tau),$$

where $a = 0, 1/q, \ldots, (q-1)/q$, $\Theta(z,\tau) = \Theta_0(z,\tau)$ is the standard theta function and $\Theta_{a,b}(z,\tau) = T_a S_b \Theta(z,\tau)$, $a \in \frac{1}{N}\mathbf{Z} \bmod 1$, $b \in \frac{1}{M}\mathbf{Z} \bmod 1$ [23] is the notation for theta functions with characteristics $\begin{bmatrix} N \\ M \end{bmatrix}$. In this basis

$$\mathcal{S}_{\pm 1}\Theta_{a,0} = \exp(\pm 2\pi i a)\Theta_{a,0}$$
$$\mathcal{T}_{\pm\frac{1}{q}}\Theta_{a,0} = \Theta_{a\pm\frac{1}{q},0}$$

Comparing these equations with (14) and (16) we obtain for parameters $\bar{\alpha}_\pm$, $\bar{\beta}_\pm$

$$\bar{\alpha}_\pm = \exp(\mp i k_1) \quad \bar{\beta}_\pm = \exp(\mp i k_2),$$

The space $\Psi(\vec{k})$ is identified with the space of theta functions with characteristic $\begin{bmatrix} q \\ 1 \end{bmatrix}$:

$$\psi_n(\vec{k}) \sim \Theta_{\frac{n}{q},0}(z,\tau)$$

The space $\text{Th}_q(\tau)$ have some analog with the space of polynomials of degree $q - 1$. The polynomial decomposition analog for the $\Theta^{(q)}$ is the following. Every $\Theta^{(q)} \in \text{Th}_q$ can be represented in the form

$$\Theta^{(q)}(z,\tau) = \gamma \exp\left(-\frac{2\pi i k z}{q}\right) \prod_{j=1}^q \Theta\left(\frac{z - z_j}{q}, \frac{\tau}{q}\right) \quad (20)$$

where $\sum_{i=1}^q z_i = k\tau + nq$, $k, n \in \mathbf{Z}$, $0 \le k, n < q$, $\gamma \in \mathbf{C}$. Then for our Hamiltonian on triangular lattice

$$\mathcal{H} = t_2 t_3 \left(e^{-i\frac{\Phi}{6}} T_{\vec{\mu}_1} + e^{i\frac{\Phi}{6}} T_{\vec{\mu}_{-1}}\right) + t_1 t_3 \left(e^{i\frac{\Phi}{6}} T_{\vec{\mu}_2} + \right.$$
$$\left. + e^{-i\frac{\Phi}{6}} T_{-\vec{\mu}_2}\right) + t_1 t_2 \left(e^{-i\frac{\Phi}{6}} T_{\vec{\mu}_3} + e^{i\frac{\Phi}{6}} T_{-\vec{\mu}_3}\right), \quad (21)$$

which was induced from the Hofstadter problem on the honeycomb lattice and the Schrödinger equation $\mathcal{H}\psi = \mathcal{E}\psi$ can be written in terms of theta functions. It is convenient to write them in terms of

$$\Theta_1(z,\tau) = \exp\left(\frac{\pi i \tau}{4} + \frac{\pi i}{2} - \pi i z\right) \Theta\left(z + \frac{1-\tau}{2}, \tau\right),$$

which obey $\Theta_1(0,\tau) = 0$. After rescaling of the arguments of $\Theta_1(z,\tau)$: $z \to qz$, $\tau \to q\tau$ and performing additional substitution: $z \to z + \frac{1+\tau}{2}$ we obtain

$$(-1)^p \beta_+ e^{i\frac{\Phi}{6} - \frac{2\pi i k p}{q}} \prod_{i=1}^q \Theta_1\left(z - z_i + \frac{p}{q}, \tau\right) + (-1)^p \beta_- e^{-i\frac{\Phi}{6} + \frac{2\pi i k p}{q}} \prod_{i=1}^q \Theta_1\left(z - z_i - \frac{p}{q}, \tau\right)$$

$$+ \alpha_+ e^{-i\frac{\Phi}{6} + \pi i \tau \frac{-q+1+2k}{q} - 2\pi i z} \prod_{i=1}^q \Theta_1\left(z - z_i - \frac{\tau}{q}, \tau\right) + \alpha_- e^{i\frac{\Phi}{6} + \pi i \tau \frac{q+1-2k}{q} + 2\pi i z} \prod_{i=1}^q \Theta_1\left(z - z_i + \frac{\tau}{q}, \tau\right)$$

$$+ (-1)^p \delta_+ e^{i\frac{\Phi}{3} + \pi i \tau \frac{1-2k+q}{q} + 2\pi i z - \frac{2\pi i k p}{q}} \prod_{i=1}^q \Theta_1\left(z - z_i + \frac{\tau+p}{q}, \tau\right) \quad (22)$$

$$+ (-1)^p \delta_- e^{i\frac{2\Phi}{3} + \pi i \tau \frac{1+2k-q}{q} - 2\pi i z + \frac{2\pi i k p}{q}} \prod_{i=1}^q \Theta_1\left(z - z_i - \frac{\tau+p}{q}, \tau\right) = (\mathcal{E} - 3) \prod_{i=1}^q \Theta_1\left(z - z_i, \tau\right)$$

where

$$\alpha_\pm = t_2 t_3 \bar{\alpha}_\pm = t_2 t_3 e^{\mp i k_1}, \; \beta_\pm = t_1 t_3 \bar{\beta}_\pm = t_1 t_3 e^{\mp i k_2}, \; \delta_\pm = t_1 t_2 \bar{\alpha}_\mp \bar{\beta}_\pm = t_1 t_2 e^{\pm i (k_1 - k_2)} \quad (23)$$

and $\mathcal{E}$ is the eigenvalue of the Hamiltonian (21) on the triangular lattice.

The roots of right and left side of (22) must coincide.

So, inserting the zeros $z = z_i$, $i = 1, \ldots, q$ of Θ_1 into the left hand side we obtain the system of q equations for the spectral parameters z_i:

$$(-1)^p \beta_+ e^{i\frac{\Phi}{6} - \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i + \frac{p}{q}, \tau\Big) + (-1)^p \beta_- e^{-i\frac{\Phi}{6} + \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i - \frac{p}{q}, \tau\Big)$$

$$+ \ \alpha_+ e^{-i\frac{\Phi}{6} + \pi i\tau \frac{-q+1+2k}{q} - 2\pi iz_j} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i - \frac{\tau}{q}, \tau\Big) + \alpha_- e^{i\frac{\Phi}{6} + \pi i\tau \frac{q+1-2k}{q} + 2\pi iz_j} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i + \frac{\tau}{q}, \tau\Big)$$

$$+ \ (-1)^p \delta_+ e^{i\frac{\Phi}{3} + \pi i\tau \frac{1-2k+q}{q} + 2\pi iz_j - \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i + \frac{\tau+p}{q}, \tau\Big) \qquad (24)$$

$$+ \ (-1)^p \delta_- e^{i\frac{2\Phi}{3} + \pi i\tau \frac{1+2k-q}{q} - 2\pi iz_j + \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\Big(z_j - z_i - \frac{\tau+p}{q}, \tau\Big) = 0$$

This is an analog of Bethe Ansatz equation for the problem under consideration, but it has no convinient form for further investigations and needs to be simplified.

By the use of the nesting procedure in the Appendix A we show that the Schrödinger equation (22) transforms to the very simple form

$$(\mathcal{E} - 3) \ = \ \bar{\gamma} \frac{\prod_{i=1}^{q} \Theta_1\big(z - z_i^{(12)}, \tau\big)}{\prod_{i=1}^{q} \Theta_1\big(z - z_i, \tau\big)}$$

$$+ \ \gamma_3 \frac{\prod_{i=1}^{q} \Theta_1\big(z - z_i^{(3)}, \tau\big)}{\prod_{i=1}^{q} \Theta_1\big(z - z_i, \tau\big)}, \qquad (25)$$

where parameters γ_3 and $\bar{\gamma}$ are defined by the formulas (A8 - A7), while the set of Bethe equations (24) is equivalent to the following larger, but much simpler set of nested Bethe equations

$$\frac{\prod_{i=1}^{q} \Theta_1(z_j^{(1)} - z_i + \frac{p}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_j^{(1)} - z_i - \frac{p}{q}, \tau)} = -\frac{\beta_-}{\beta_+} e^{-i\frac{\Phi}{3} + \frac{4\pi ikp}{q}}, \qquad (26)$$

$$\frac{\prod_{i=1}^{q} \Theta_1(z_j^{(2)} - z_i - \frac{\tau}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_j^{(2)} - z_i + \frac{\tau}{q}, \tau)} =$$

$$= -\frac{\alpha_-}{\alpha_+} e^{i\frac{\Phi}{3} + 2\pi i\tau \frac{q-2k}{q} + 4\pi iz_j^{(2)}}, \qquad (27)$$

$$\frac{\prod_{i=1}^{q} \Theta_1(z_j^{(3)} - z_i + \frac{\tau+p}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_j^{(3)} - z_i - \frac{\tau+p}{q}, \tau)} =$$

$$= -\frac{\delta_-}{\delta_+} e^{i\Phi(\frac{1}{3} + 2k) - 2\pi i\tau \frac{q-2k}{q} - 4\pi iz_j^{(3)}}, \qquad (28)$$

$$e^{i\frac{\Phi}{3} - \pi i\tau \frac{3-q+2k}{q} - \frac{2\pi ikp}{q} + 2\pi iz_1} \frac{\prod_{i=1}^{q} \Theta_1(z_j^{(12)} - z_i^{(1)}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_j^{(12)} - z_i^{(2)}, \tau)}$$

$$= (-1)^{p+1} \frac{\alpha_+}{\beta_+} \frac{\prod_{i=1}^{q} \Theta_1(z_1 - z_i - \frac{2\tau}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_1 - z_i^{(2)} - \frac{\tau}{q}, \tau)}$$

$$\cdot \frac{\prod_{i=1}^{q} \Theta_1(z_1 - z_i^{(1)} + \frac{p}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_1 - z_i + \frac{2p}{q}, \tau)} \qquad (29)$$

and

$$e^{-i\frac{7\Phi}{6} - \pi i\tau \frac{3-2k+q}{q} - 2\pi iz_1} \frac{\prod_{i=1}^{q} \Theta_1(z_j - z_i^{(12)}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_j - z_i^{(3)}, \tau)}$$

$$= -\frac{\delta_+}{\beta_+} \frac{\prod_{i=1}^{q} \Theta_1(z_1 - z_i + \frac{2(p+\tau)}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_1 - z_i^{(3)} + \frac{(p+\tau)}{q}, \tau)}$$

$$\cdot \frac{\prod_{i=1}^{q} \Theta_1(z_1^{(2)} - z_i^{(12)}, \tau) \cdot \Theta_1(z_1 - z_i^{(1)} + \frac{p}{q}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_1^{(2)} - z_i^{(1)}, \tau) \cdot \Theta_1(z_1 - z_i + \frac{2p}{q}, \tau)}. \qquad (30)$$

We have obtained a set of five equations for the set of five parameters $z_i, z_i^{(1)}, z_i^{(2)}, z_i^{(3)}, z_i^{(12)}$, $i = 1, 2, \cdots q$. The parameters $\alpha_\pm$, $\beta_\pm$ and $\delta_\pm$ depend on momenta $\vec{k}$. Therefore the solutions $z_i, z_i^{(1)}, z_i^{(2)}, z_i^{(3)}, z_i^{(12)}$, $i = 1, 2, \cdots q$ of the equations (26)-(30) depend on the momenta too.

Both terms in the sum on the left hand side of (25) are elliptic functions, i.e. 2-periodic, in the fundamental domain with vertexes $(0, \tau, 1, 1 + \tau)$, and each of them has precisely q zeroes and q poles there. Let us suggest that all poles are single. It is known that such an elliptic function can be written in the form

$$f(z) = C + \sum_{i=1}^{q} A_i \zeta(z - z_i), \qquad (31)$$

where $\zeta(z)$ is a ζ-function of Weierstrass for the same periodicity domain. Formula (31) is an analog of decomposition of rational functions with single poles in terms of functions $\frac{1}{z - z_i}$.

Applying (31) like decomposition for elliptic functions to the expression (25) for the energy one can verify that the conditions for the annulation of the residues of the poles precisely are coinciding with (26)-(30) (or equivalently (24)). Therefore the energy E is independent of z constant function C in this decomposition and one can find (see formulas (A9) and (A6) in the Appendix)

$$\mathcal{E} = 3 + (-1)^p \delta_+ e^{i\Phi(\frac{4}{3} - k) + \pi i\tau \frac{3-2k+q}{q} + 2\pi iz_1} \qquad (32)$$

$$\cdot \frac{\prod_{i=1}^{q} \Theta_1(z_1 - z_i + \frac{2(p+\tau)}{q}, \tau) \cdot \Theta_1(z_1^{(12)} - z_i^{(3)}, \tau)}{\prod_{i=1}^{q} \Theta_1(z_1 - z_i^{(3)} + \frac{(p+\tau)}{q}, \tau) \cdot \Theta_1(z_1^{(12)} - z_i, \tau)},$$

This expression defines the spectrum of the states in the Hofstadter problem on the honeycomb lattice as a function of momenta because the solutions z_i, $z_i^{(1)}$, $z_i^{(2)}$, $z_i^{(3)}$, $z_i^{(12)}$, $i = 1, 2, \cdots q$, of the equations (26-30) are momentum dependent.

This procedure reminds Nested Bethe Ansatz where we have a nested chain of Bethe equations (see as an example the set of nested Bethe equations in t-J-[24] and staggered t-J models[26]).

Although we have increased the amount of Bethe equations (26)-(30) together with the amount of parameters to be defined, each of the equations (26- 30) now is much simpler and allows to apply the technique of Thermodynamic Bethe Ansatz[27,28] for their investigation. After taking logarithm of the left and right hand sides of these equations, one can reduce them to the set of integral equations in the most interesting limit of $q \to \infty$, necessary in the irrational flux case. This fact justifies the procedure of nested Bethe Ansatz.

Nested Bethe equations (26- 30) now have a form, which allows to investigate them further by use of Thermodynamic Bethe Ansatz. One can take logarithm of the left and right hand sides and transform it to integral equation in the thermodynamic limit $q \to \infty$.

APPENDIX A: NESTING OF BETHE ANSATZ EQUATIONS

Since any term in the linear space of functions $\Theta^{(q)} \in \mathrm{Th}_q$ can be represented as a product of theta functions (20) with some particular choice of zeros at $z_j, j = 1, 2, \cdots q$, the sum of the first and second products of theta functions in the expression (22) can be represented as

$$(-1)^p \beta_+ e^{i\frac{\Phi}{6} - \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\left(z - z_i + \frac{p}{q}, \tau\right) +$$

$$+ (-1)^p \beta_- e^{-i\frac{\Phi}{6} + \frac{2\pi ikp}{q}} \prod_{i=1}^{q} \Theta_1\left(z - z_i - \frac{p}{q}, \tau\right)$$

$$= \gamma_1 \prod_{i=1}^{q} \Theta_1\left(z - z_i^{(1)}, \tau\right). \quad \text{(A1)}$$

Consequently, the sum of the third and forth terms in (22) (respectively the sum of fifth and sixth terms) define

$$\alpha_+ e^{-i\frac{\Phi}{6} + \pi i\tau \frac{-q+1+2k}{q} - 2\pi iz} \prod_{i=1}^{q} \Theta_1\left(z - z_i - \frac{\tau}{q}, \tau\right) +$$

$$+ \alpha_- e^{i\frac{\Phi}{6} + \pi i\tau \frac{q+1-2k}{q} + 2\pi iz} \prod_{i=1}^{q} \Theta_1\left(z - z_i + \frac{\tau}{q}, \tau\right)$$

$$= \gamma_2 \prod_{i=1}^{q} \Theta_1\left(z - z_i^{(2)}, \tau\right), \quad \text{(A2)}$$

$$(-1)^p \delta_+ e^{i\frac{\Phi}{3} + \pi i\tau \frac{1-2k+q}{q} + 2\pi iz - \frac{2\pi ikp}{q}}$$

$$\cdot \prod_{i=1}^{q} \Theta_1\left(z - z_i + \frac{\tau + p}{q}, \tau\right)$$

$$+ (-1)^p \delta_- e^{i\frac{2\Phi}{3} + \pi i\tau \frac{1+2k-q}{q} - 2\pi iz + \frac{2\pi ikp}{q}}$$

$$\cdot \prod_{i=1}^{q} \Theta_1\left(z - z_i - \frac{\tau + p}{q}, \tau\right) \quad \text{(A3)}$$

$$= \gamma_3 \prod_{i=1}^{q} \Theta_1\left(z - z_i^{(3)}, \tau\right),$$

Right hand sides of the equations (A1), (A2) and (A3) become zero at $z = z_j^{(1)}$, $z = z_j^{(2)}$, $z = z_j^{(3)}$, $j = 1, 2 \cdots q$, respectively, which imposes restrictions on the parameters $z_i, z_i^{(1)}, z_i^{(2)}, z_i^{(3)}, i = 1, 2, \cdots q$ in the form of Bethe equations (26 - 28).

The parameters γ_1, γ_2 and γ_3 are z independent constants, as follows from the arguments presented in the previous section by use of representation of elliptic functions via Weierstrass ζ function (31). Therefore they can be fixed by inserting any convenient value z into the corresponding equation. By putting values $z = z_1 + p/q$, $z = z_1 - \tau/q$, $z = z_1 + p/q + \tau/q$ into the equations (A1), (A2) and (A3) respectively, which annulate one of terms in the left hand sides, one can solve this equations for γ's and obtain

$$\gamma_1 = (-1)^p \beta_+ e^{i\Phi(\frac{1}{6} - k)}$$
$$\cdot \frac{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i + \frac{2p}{q}, \tau\right)}{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i^{(1)} + \frac{p}{q}, \tau\right)} \quad \text{(A4)}$$

$$\gamma_2 = \alpha_+ e^{-i\frac{\Phi}{6} + \pi i\tau \frac{3-q+2k}{q} - 2\pi iz_1}$$
$$\cdot \frac{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i - \frac{2\tau}{q}, \tau\right)}{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i^{(2)} - \frac{\tau}{q}, \tau\right)} \quad \text{(A5)}$$

$$\gamma_3 = (-1)^p \delta_+ e^{i\Phi(\frac{4}{3} - k) + \pi i\tau \frac{3-2k+q}{q} + 2\pi iz_1}$$
$$\cdot \frac{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i + \frac{2(\tau + p)}{q}, \tau\right)}{\prod_{i=1}^{q} \Theta_1\left(z_1 - z_i^{(3)} + \frac{(\tau + p)}{q}, \tau\right)} \quad \text{(A6)}$$

Now let us continue further this procedure, which reminds nesting Bethe Ansatz in the some of integrable models[24,25,26], and transform the sum of right hand sides of (A1) and (A2) into the product of theta functions with the zeros at the new positions $z_i^{(12)}$

$$\gamma_1 \prod_{i=1}^{q} \Theta_1(z - z_i^{(1)}, \tau) + \gamma_2 \prod_{i=1}^{q} \Theta_1(z - z_i^{(2)}, \tau)$$
$$= \bar{\gamma} \prod_{i=1}^{q} \Theta_1(z - z_i^{(12)}, \tau) \qquad (A7)$$

Now again, one can insert the value $z = z_1^{(2)}$ into the equation above and obtain

$$\bar{\gamma} = \gamma_1 \frac{\prod_{i=1}^{q} \Theta_1\big(z_1^{(2)} - z_i^{(1)}, \tau\big)}{\prod_{i=1}^{q} \Theta_1\big(z_1^{(2)} - z_i^{(12)}, \tau\big)}, \qquad (A8)$$

where γ_1 defined by the expression in (A4). The spectral parameters $z_i^{(12)}$ are the zeros of the left(and, therefore, of the right) hand side of the equation (A7). Hence, by use of (A4) and (A8) we obtain Bethe equations (29) for $z^{(12)}$.

With this reformulations of the sums of theta functions we bring Schrödinger equation (22) for the eigenenergy $\mathcal{E}$ to the very simple form

$$(\mathcal{E} - 3) \prod_{i=1}^{q} \Theta_1(z - z_i, \tau) = \bar{\gamma} \prod_{i=1}^{q} \Theta_1\big(z - z_i^{(12)}, \tau\big)$$
$$+ \gamma_3 \prod_{i=1}^{q} \Theta_1\big(z - z_i^{(3)}, \tau\big) \quad (A9)$$

By inserting $z = z_1^{(12)}$ into this expression (the first term on the right hand side becomes zero) and using (A6) for γ_3 one can obtain simple expression (32) for the energy. This complets Bethe Ansatz solution of the Hofstadter problem on the honeycomb lattice.

The set of nested Bethe equations (26 - 30), are equivalent to the single set (24). We have more equations in (26 - 30) than in (24), but their advantage is in their simplicity. Each of mentioned equations contains only two product of theta functions, which will allow to implement the technique of Thermodynamic Bethe Ansatz[27,28] for further investigations.

[*] Electronic address: kohmoto@issp.u-tokyo.ac.jp
[†] Electronic address: sedrak@alf.nbi.dk
[1] K.S. Novoselov et al., Nature **438**, 197 (2005).
[2] Y. Zheng, Y.-W. Tan, H.L. Shtormer, and P. Kim, Nature **438**, 201 (2005).
[3] R. Jackiw, Phys. Rev. D 27, 2375 (1984).
[4] G.W. Semenoff, Phys.Rev.Lett. **53**,2449 (1984).
[5] F.D.M. Haldane, Phys.Rev.Lett. /bf 61, 2015 (1988).
[6] Y. Zheng and T. Ando, Phys.Rev. B 65, 245420 (2002).
[7] V.P. Gusinin and S.G. Sharapov, Phys.Rev.Lett./bf 95 146801 (2005).
[8] N.M.R. Peres, F. Guinea and A.H.C. Neto, cond-mat/0506709.
[9] M.Ya. Azbel, Sov. Phys. JETP /bf 19, 634 (1964),
[10] J. Zak, Phys.Rev. 134, 1602 (1964).
[11] G.H. Wannier, Phys.Status Solidi, 88,757 (1978).
[12] D.R. Hofstadter, Phys. Rev.**B 14**,2239 (1976).
[13] D.J. Thouless, M. Kohmoto, M.P. Nightingale and M. Den Nijs, Phys. Rev. Lett. **49**, 405408 (1982).
[14] M. Kohmoto, Ann. Phys. **160**, 343 (1985).
[15] M. Kohmoto, Phys. Rev. B 39, 11943 (1989),
X.G. Wen, A. Zee, Nucl. Phys. **B 326**, 619 (1989).
[16] H. Hiramoto and M. Kohmoto, Phys.Rev. **B40**, 8225 (1989).
[17] Y. Hatsugai and M. Kohmoto, Phys. Rev. B 42, 8282 (1990),
J.H. Han, D.J. Thouless, H. Hiramoto, and M. Kohmoto,
Phys.Rev. **B 50**, 11365 (1994),
I. Chang, K. Ikezawa and M. Kohmoto, Phys. Rev. B 55, 12972 (1997).
[18] Y. Hasegawa, Y. Hatsugai, M. Kohmoto and G. Montambaux, Phys.Rev. B 41, 9174 (1990).
[19] P.W.Wiegmann and A.V.Zabrodin, Phys.Rev.Lett. **72**,1890 (1994),
A.G. Abanov, J.C. Talstra and P.W.Wiegmann, Phys.Rev.Lett. 81, 2112 (1998).
[20] R.M.Kashaev L.D.Faddeev, Commun.Math.Phys. **169**,181 (1995).
[21] T. Hakobyan, A. Sedrakyan. Mod. Phys. Lett. **A10**, 495 (1995).
[22] Y. Hatsugai, M. Kohmoto and Y.-S. Wu, Phys. Rev. B 53, 9697 (1996).
[23] D. Mamford, *Tata lectures on theta*, Progress in Mathematics, 28,1983.
[24] B. Sutherland, Phys. Rev. **B 12**, 3795 (1975).
[25] F. Essler, V. Korepin, Phys. Rev. **B 46**, 9247 (1992).
[26] T. Sedrakyan, Nucl. Phys. **B 608 [FS]**, 557 (2001),
J.Ambjorn, D. Arnaudon, A. Sedrakyan, T.Sedrakyan and P.Sorba, J.Phys.A: Math. Gen. **34**, 5887 (2001).
[27] M. Takahashi,Prog.Theor.Phys.**46**,401 (1971).
[28] M. Gaudin, Phys.Rev.Lett. **26**,1301 (1971).
[29] K. Ino, M. Kohmoto, cond-mat/0512562.